Calculation Example: Futures Arbitrage

Question: A non-dividend paying stock is currently priced at $100. A one year futures contract on this stock currently sells at a futures price of $105. The risk free rate is 10% pa continuously compounded. Construct an ideal risk free arbitrage using an arbitrage table.

Arbitrage Table

	Cash flows	
Action	**t=0**	**t=T**
? future	0	?
? stock	100	?
? bond	?	?
Total	?	?

Answer: The one-year futures contract should have a theoretical price ($F_{0,1,theoretical}$) equal to:

$$F_{0,T,theoretical} = (S_0 - D_0).e^{r.T}$$

$$F_{0,1,theoretical} = (100 - 0) \times e^{0.1 \times 1} = 110.5170918$$

The actual price is lower:

$$F_{0,1,actual} = 105$$

Therefore either the futures contract is under-priced and should be bought, or the stock is over-priced and should be sold, or both!

To hedge the risk of the long future, we can short a 'synthetic future'. The synthetic short future can be constructed by remembering the futures valuation equation:

$$V_{T,SF} = -V_{T,LF} = -(S_T - K_T) = \boldsymbol{K_T} - \boldsymbol{S_T}$$

Looking at these $(\boldsymbol{K_T} - \boldsymbol{S_T})$ as cash flows at maturity:

- $-S_T$ is a negative cash flow at maturity, which means we have to pay money at the end, which must be from short-selling the stock so at the end we have to buy the stock back to return it to the stock lender. Since the final stock price is unknown, we just call this S_T.

- K_T is a positive cash flow at maturity, and it's independent of the stock price S_T. This means we'll receive money at maturity, say from a bond. Therefore we must have bought a bond at the start (long bond or lending) because we're being paid back at the end.

Futures Arbitrage Table Working

Cash flows

Action	t=0	t=T
Long future	0	$S_T - F_{0,T,actual}$
Short stock	S_0	$-S_T$
Long bond	$-F_{0,T,actual}/e^{r.T}$	$F_{0,T,actual}$
Total	$0 + S_0 - F_{0,T,actual}/e^{r.T}$	0

Futures Arbitrage Table Results

Cash flows

Action	t=0	t=1
Long future	0	$S_1 - 105$
Short stock	100	$-S_1$
Long bond	-95.00792889	105
Total	4.992071106	0

Notice that the present value of the physical (not synthetic) mis-priced future is the same as our present value of total cash flows above:

$$V_{0,SF\ physical} = V_{T,SF\ physical}/e^{r.T}$$

$$= (K_T - S_T)/e^{r.T}$$

$$= (K_T - (S_0 - D_0).e^{r.T})/e^{r.T} \quad or \quad = \left(K_T - S_0.e^{(r-q).T}\right)/e^{r.T}$$

$$= (105 - (100 - 0) \times e^{0.1 \times 1})/e^{0.1 \times 1}$$

$$= 4.992071106$$